The While language

x : variable
S : statement
a : arithmetic expression
b : boolean expression

$$\begin{array}{lll} S & ::= & x := a \mid \mathsf{skip} \mid S_1; S_2 \mid \\ & & \mathsf{if}\ b\ \mathsf{then}\ S_1\ \mathsf{else}\ S_2 \\ & & \mathsf{while}\ b\ \mathsf{do}\ S\ \mathsf{od} \end{array}$$

Syntactic categories

- Numbers

$$n \in \mathbf{Num} = \{0, \cdots, 9\}^+$$

- Variables

$$x \in \mathbf{Var}$$

- Arithmetic expressions

$$\begin{array}{lll} a & \in & \mathbf{Aexp} \\ a & := & n \mid x \mid a_1 + a_2 \mid a_1 * a_2 \mid a_1 - a_2 \end{array}$$

- Boolean expressions

$$\begin{array}{lll} b & \in & \mathbf{Bexp} \\ b & := & \mathsf{true} \mid \mathsf{false} \mid a_1 = a_2 \mid a_1 \leq a_2 \mid \neg b \mid b_1 \wedge b_2 \end{array}$$

- Statements

$$\begin{array}{lll} S & \in & \mathbf{Stm} \\ S & ::= & x := a \mid \mathsf{skip} \mid S_1; S_2 \mid \\ & & \mathsf{if}\ b\ \mathsf{then}\ S_1\ \mathsf{else}\ S_2 \\ & & \mathsf{while}\ b\ \mathsf{do}\ S\ \mathsf{od} \end{array}$$

Semantic functions

- Digits : integers

$$\mathcal{N} : \mathbf{Num} \rightarrow \mathbb{Z}$$

- Arithmetic expressions: for each state, a value in $\mathbb{Z}$

$$\mathcal{A} : \mathbf{Aexp} \rightarrow (\mathbf{State} \rightarrow \mathbb{Z})$$

- Boolean expressions: for each state, a value in $\mathbb{B}$

$$\mathcal{B} : \mathbf{Bexp} \rightarrow (\mathbf{State} \rightarrow \mathbb{B})$$

- Statements:

$$\mathcal{S} : \mathbf{Stm} \rightarrow (\mathbf{State} \xrightarrow{part.} \mathbf{State})$$

Arithmetic expressions semantics

$$\mathcal{N}(n_1 \cdots n_k) = \Sigma_{i=1}^{k} n_i \cdot 10^{k-i}$$

$$\begin{array}{ll} \mathcal{A}[n]\sigma & = \mathcal{N}[n] \\ \mathcal{A}[x]\sigma & = \sigma(x) \\ \mathcal{A}[a_1 + a_2]\sigma & = \mathcal{A}[a_1]\sigma +_I \mathcal{A}[a_2]\sigma \\ \mathcal{A}[a_1 * a_2]\sigma & = \mathcal{A}[a_1]\sigma *_I \mathcal{A}[a_2]\sigma \\ \mathcal{A}[a_1 - a_2]\sigma & = \mathcal{A}[a_1]\sigma -_I \mathcal{A}[a_2]\sigma \end{array}$$

The semantics of arithmetic expressions is inductively defined over their structure. It is a compositional semantics.

Natural semantics

- $$(x := a, \sigma) \rightarrow \sigma[x \mapsto \mathcal{A}[a]\sigma]$$

- $$(\mathsf{skip}, \sigma) \rightarrow \sigma$$

- $$\frac{(S_1, \sigma) \rightarrow \sigma', \quad (S_2, \sigma') \rightarrow \sigma''}{(S_1; S_2, \sigma) \rightarrow \sigma''}$$

- If $\mathcal{B}[b]\sigma = \mathbf{tt}$ then

$$\frac{(S_1, \sigma) \rightarrow \sigma'}{(\mathsf{if}\ b\ \mathsf{then}\ S_1\ \mathsf{else}\ S_2, \sigma) \rightarrow \sigma'}$$

- If $\mathcal{B}[b]\sigma = \mathbf{ff}$ then

$$\frac{(S_2, \sigma) \rightarrow \sigma'}{(\mathsf{if}\ b\ \mathsf{then}\ S_1\ \mathsf{else}\ S_2, \sigma) \rightarrow \sigma'}$$

- If $\mathcal{B}[b]\sigma = \mathbf{tt}$ then

$$\frac{(S, \sigma) \rightarrow \sigma', \quad (\mathsf{while}\ b\ \ \mathsf{do}\ S\ \mathsf{od}\ , \sigma') \rightarrow \sigma''}{(\mathsf{while}\ b\ \ \mathsf{do}\ S\ \mathsf{od}\ , \sigma) \rightarrow \sigma''}$$

- If $\mathcal{B}[b]\sigma = \mathbf{tt}$ then

$$\frac{(S, \sigma) \rightarrow \sigma', \quad (\mathsf{while}\ b\ \ \mathsf{do}\ S\ \mathsf{od}\ , \sigma') \rightarrow \sigma''}{(\mathsf{while}\ b\ \ \mathsf{do}\ S\ \mathsf{od}\ , \sigma) \rightarrow \sigma''}$$

- If $\mathcal{B}[b]\sigma = \mathbf{ff}$ then

$$(\mathsf{while}\ b\ \ \mathsf{do}\ S\ \mathsf{od}\ , \sigma) \rightarrow \sigma$$