Structural induction

Principle

To prove that a property holds by structural induction, you must check the following things.

- Prove that the property holds for all the basic elements, atoms, of the set.
- Prove that the property holds for all the composite elements, those created by the application of rules. To do that, assume the immediate components of the element verify the property (what we call the induction hypothesis) and prove that the composite element verifies the property too.

Example

The While language is defined as follows:

$$\begin{array}{lcl} S & \in & \mathbf{Stm} \\ S & ::= & x := a \mid \mathsf{skip} \mid S_1 ; S_2 \mid \\ & & \mathsf{if}\ b\ \mathsf{then}\ S_1\ \mathsf{else}\ S_2 \\ & & \mathsf{while}\ b\ \ \mathsf{do}\ S\ \mathsf{od} \end{array}$$

To show that a property P holds for every statement of the While language, we must show:

- Atoms: check that $x := a$ and skip verify P.
- Composite elements: the last rule applied to build a composite element can be ";", "*if*" or "*while*".
  1. composite element of the form $S_1; S_2$: its immediate components are S_1 and S_2, the induction hypothesis we can make is that P is true for S_1 and for S_2. We have to show that P is true for $S_1; S_2$.
  2. composite element of the form if b then S_1 else S_2: its immediate components are S_1 and S_2, the induction hypothesis we can make is that P is true for S_1 and for S_2. We have to show that P is true for if b then S_1 else S_2.
  3. composite element of the form while b do S od : one immediate component S. The induction hypothesis we make is that P holds for S, and we must show that P holds for while b do S od .

Induction on the shape of a derivation tree

Principle

We want to prove that a property P is true by induction on the shape of a derivation tree.

- We show that the property holds for 'one-rule' derivation trees, that is to say, trees just consisting of an axiom.
- We show that the property holds for composite trees. To do so, for each rule R, we consider a composite tree for which the last rule applied to build it was R, and we must show that the property P is true for the composite tree. The induction hypothesis we make is immediate components - that is to say, subtrees, or premises of the rule - of the composite elements verify P.

Example

We consider the case of the system of rules defined for natural operational semantics.
To show that a property P holds for all $< S, \sigma > \rightarrow \sigma'$, we identify $< S, \sigma > \rightarrow \sigma'$ with the tree concluding in this statement. We have to check the following things to have a complete proof by induction.

- We check that the property P holds for axioms. Axioms of this system are the skip rule and the assignment rule.

- We check that the property holds for an arbitrary composite tree. We have three possibility of rules that were applied last to build this tree.

  1. The sequence rule was applied last.

  $$\frac{(S_1, \sigma) \rightarrow \sigma', \quad (S_2, \sigma') \rightarrow \sigma''}{(S_1; S_2, \sigma) \rightarrow \sigma''}$$

     The application of the rules puts to use two strict subtrees of our composite tree, for which we can assume the induction hypothesis holds (i.e. we assume P is true for trees $(S_1, \sigma) \rightarrow \sigma'$ and $(S_2, \sigma') \rightarrow \sigma''$). We use it to prove P is true for the conclusion of our composite tree.

  2. The if rule was applied last. Two cases can arise. Either rule if-true or rule if-false was applied.

     – Suppose rule 'if-true' was applied. This is assuming $\mathcal{B}[b]\sigma = \mathbf{tt}$, and the composite tree has the form:

     $$\frac{(S_1, \sigma) \rightarrow \sigma'}{(\mathsf{if}\ b\ \mathsf{then}\ S_1\ \mathsf{else}\ S_2, \sigma) \rightarrow \sigma'}$$

       The premise of the rule is the subtree $(S_1, \sigma) \rightarrow \sigma'$. The induction hypothesis we make is that $(S_1, \sigma) \rightarrow \sigma'$ verifies P, and we show that the conclusion of the composite tree verifies P.

     – Suppose rule 'if-false' was applied. This is assuming $\mathcal{B}[b]\sigma = \mathbf{ff}$, and the composite tree has the form:

     $$\frac{(S_2, \sigma) \rightarrow \sigma'}{(\mathsf{if}\ b\ \mathsf{then}\ S_1\ \mathsf{else}\ S_2, \sigma) \rightarrow \sigma'}$$

       The premise of the rule is the subtree $(S_2, \sigma) \rightarrow \sigma'$. The induction hypothesis we make is that $(S_2, \sigma) \rightarrow \sigma'$ verifies P, and we show that the conclusion of the composite tree verifies P.

  3. While rule