UFR-IMAG
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Series 1

Exercise 4

We wish to add the following statement to the **While** language:

$\textsf{repeat } S \textsf{ until } b$

The rules we add to the rules of natural semantics are:

- If $\mathcal{B}[b]\sigma' = \mathbf{ff}$ then

$$\frac{(S,\sigma) \to \sigma' \quad (\textsf{repeat } S \textsf{ until } b, \sigma') \to \sigma''}{(\textsf{repeat } S \textsf{ until } b, \sigma) \to \sigma''}$$

- If $\mathcal{B}[b]\sigma' = \mathbf{tt}$ then

$$\frac{(S,\sigma) \to \sigma'}{(\textsf{repeat } S \textsf{ until } b, \sigma) \to \sigma'}$$

Indeed, the meaning we want to give to this command is that we first perform S and then, according to whether b is true, we re-enter the repeat command or we stop.

Semantic equivalence proof
We prove that

- $\textsf{repeat } S \textsf{ until } b$
- and $S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b)$.

are semantically equivalent.

To do this, we have to prove that for any states σ, σ' we have that $(\textsf{repeat } S \textsf{ until } b, \sigma) \to \sigma'$ iff $(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'$.

We first prove the $\Rightarrow$ implication. We assume $(\textsf{repeat } S \textsf{ until } b, \sigma) \to \sigma'$ and have to prove $(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'$. Assuming $(\textsf{repeat } S \textsf{ until } b, \sigma) \to \sigma'$ is assuming that there exists a derivation tree T whose conclusion is this statement. Two cases can arise:

- the tree T can be the following:

$$\frac{(S,\sigma) \to \sigma_1 \quad (\textsf{repeat } S \textsf{ until } b, \sigma_1) \to \sigma'}{(\textsf{repeat } S \textsf{ until } b, \sigma) \to \sigma'}$$

  In this case, we know that σ_1 exists and that $\mathcal{B}[b]\sigma_1 = \mathbf{ff}$.
  We are searching for a tree T' whose conclusion is $(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'$. The program is the sequence of S and an if command. Such a tree T' would *necessary look like*:

$$\frac{(S,\sigma) \to \sigma_2 \qquad \dfrac{?}{(\textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma_2) \to \sigma'}}{(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'}$$

for some candidate σ_2 we have to exhibit. If we look at the tree T, we see that we know $(S,\sigma) \to \sigma_1$. Hence we choose $\sigma_2 = \sigma_1$. Our tree T' becomes:

$$\dfrac{(S,\sigma)\to\sigma_1 \qquad \dfrac{?}{(\textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma_1) \to \sigma'}}{(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'}$$

We still have to replace ?, which we can do because we know that $\mathcal{B}[b]\sigma_1 = \mathbf{ff}$. Hence, we apply the if-false rule to derive a tree for $(\textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma_1) \to \sigma'$. T' thus looks like:

$$\dfrac{(S,\sigma)\to\sigma_1 \qquad \dfrac{(\textsf{repeat } S \textsf{ until } b, \sigma_1) \to \sigma_3}{(\textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma_1) \to \sigma'}}{(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'}$$

for some σ_3 we have to find. Looking at T, we see that $\sigma_3 = \sigma'$ fits.

$$\dfrac{(S,\sigma)\to\sigma_1 \qquad \dfrac{(\textsf{repeat } S \textsf{ until } b, \sigma_1) \to \sigma'}{(\textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma_1) \to \sigma'}}{(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'}$$

is the derivation tree we are looking for. QED

- the tree T can be the following:

$$\dfrac{(S,\sigma)\to\sigma'}{(\textsf{repeat } S \textsf{ until } b, \sigma) \to \sigma'}$$

In this case, we know that σ' exists and that $\mathcal{B}[b]\sigma' = \mathbf{tt}$.
We are searching for a tree T' whose conclusion is $(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'$. The program is the sequence of S and an if command. Such a tree T' would *necessary look like*:

$$\dfrac{(S,\sigma)\to\sigma_1 \qquad \dfrac{?}{(\textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma_1) \to \sigma'}}{(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'}$$

for some candidate σ_1 we have to exhibit. If we look at the tree T, we see that we know $(S,\sigma) \to \sigma'$. Hence we choose $\sigma_1 = \sigma'$. Our tree T' becomes:

$$\dfrac{(S,\sigma)\to\sigma' \qquad \dfrac{?}{(\textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma') \to \sigma'}}{(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'}$$

We still have to replace ?, which we can do because we know that $\mathcal{B}[b]\sigma' = \mathbf{tt}$. Hence, we apply the if-false rule to derive a tree for $(\textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma') \to \sigma'$. T' thus looks like:

$$\dfrac{(S,\sigma)\to\sigma' \qquad \dfrac{(\textsf{skip}, \sigma') \to \sigma'}{(\textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma') \to \sigma'}}{(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'}$$

It is the derivation tree we are looking for. QED

We then prove the $\Leftarrow$ implication. We assume $(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'$ and have to prove $(\textsf{repeat } S \textsf{ until } b, \sigma) \to \sigma'$. Our assumption yields the existence of a derivation tree T whose conclusion is $(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'$. It necessarily looks like:

$$\cfrac{(S,\sigma) \to \sigma_1 \qquad \cfrac{?}{(\textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma_1) \to \sigma'}}{(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'}$$

with an actual state σ_1. '?' depends on the truth value of b in state σ_1.

Two cases can arise:

- if $\mathcal{B}[b]\sigma_1 = \mathbf{ff}$, we know that T is the following tree:

$$\cfrac{(S,\sigma) \to \sigma_1 \qquad \cfrac{(\textsf{repeat } S \textsf{ until } b, \sigma_1) \to \sigma'}{(\textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma_1) \to \sigma'}}{(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'}$$

  We want to build a tree T' whose conclusion is $(\textsf{repeat } S \textsf{ until } b, \sigma) \to \sigma'$. Such a tree necessarily ends with the application of one of the rules for the repeat command. We know that: $(S,\sigma) \to \sigma_1$, $(\textsf{repeat } S \textsf{ until } b, \sigma_1) \to \sigma'$, and $\mathcal{B}[b]\sigma_1 = \mathbf{ff}$. Hence, it is the repeat-true rule we use to build T':

$$\cfrac{(S,\sigma) \to \sigma_1 \qquad (\textsf{repeat } S \textsf{ until } b, \sigma_1) \to \sigma'}{(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'}$$

- Similarly, if $\mathcal{B}[b]\sigma_1 = \mathbf{tt}$, we know that T is the following tree:

$$\cfrac{(S,\sigma) \to \sigma_1 \qquad \cfrac{(\textsf{skip}, \sigma_1) \to \sigma'}{(\textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma_1) \to \sigma'}}{(S; \textsf{if } b \textsf{ then skip else } (\textsf{repeat } S \textsf{ until } b), \sigma) \to \sigma'}$$

  Moreover, according to the skip rule, $\sigma_1 = \sigma'$.
  We want to build a tree T' whose conclusion is $(\textsf{repeat } S \textsf{ until } b, \sigma) \to \sigma'$. Such a tree necessarily ends with the application of one of the rules for the repeat command. We know that $(S,\sigma) \to \sigma'$, and $\mathcal{B}[b]\sigma' = \mathcal{B}[b]\sigma_1 = \mathbf{tt}$. So we can build T' as follows:

$$\cfrac{(S,\sigma) \to \sigma'}{(\textsf{repeat } S \textsf{ until } b, \sigma) \to \sigma'}$$